Backlund transformations for the Schrodinger equation with a spectral dependence in the potential

1984 J. Phys. A: Math. Gen. 17 3151

(http://iopscience.iop.org/0305-4470/17/16/017)

J. Phys. A: Math. Gen. 17 (1984) 3151-3167. Printed in Great Britain

Bäcklund transformations for the Schrödinger equation with a spectral dependence in the potential

M Boiti, C Laddomada and F Pempinelli

Dipartimento di Fisica, Università di Lecce, Lecce, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Bari, Bari, Italy

Received 4 April 1984

Abstract. The Schrödinger equation on the line with a potential depending linearly on the spectral parameter is considered. It is shown that this spectral problem admits one-parameter (called elementary) Bäcklund transformations and two-parameter (called full) Bäcklund transformations. The transformations of the spectral data under all these Bäcklund transformations (BTs) are explicitly computed. Different nonlinear superposition formulae for the different BTs considered are given and used to obtain soliton solutions of the evolution equations associated with the spectral problem.

1. Introduction

Let us consider the Schrödinger equation on the line

$$-y_{xx}+(P+\lambda Q)y=\lambda^2 y \tag{1.1}$$

with a potential $P+\lambda Q$ depending linearly on the spectral parameter λ and P, Q defined on the whole real axis and decaying sufficiently fast for $|x|\to\infty$.

The direct and inverse problems for this equation have been solved by Jaulent and Jean (1976a, b). Subsequently Jaulent and Miodek (1976) found a hierarchy of infinitely many, so-called, soliton equations whose Cauchy problems can be linearised by the use of the Inverse Scattering Transform, or Spectral Transform (ST), associated with (1.1) (Ablowitz and Segur 1981, Calogero and Degasperis 1982).

This ST is also useful in the solution of the inverse problem which occurs in various fields of physics (transmission lines theory, electromagnetism, elasticity theory, geophysics) in which the inhomogeneous media are absorbing (Jaulent 1976).

In the last decade the relevance of the Bäcklund transformations (BT) has been stressed in the study of the soliton equations and, specifically, in finding explicitly their soliton solutions.

Recently, Sabatier (1983a) showed that the BTs can be used to 'explore accurately' both the space of the spectral data and of the potentials. In particular (Sabatier 1983b, c), one can study possible bifurcations and ambiguities in the solution of the inverse problem, which are of physical interest.

Therefore, both from the point of view of those who are interested in soliton equations and of those interested in solving the inverse problem, we are encouraged to study the BTs for the Jaulent–Jean–Miodek (JJM) spectral problem.

A two-parameter BT has already been found by Laddomada and Tu (1982).

In this paper we show that the x-component of this BT can be explicitly integrated once to yield two equations which are pure differential with respect to $\int_x^\infty Q\,dx$ and purely algebraic with respect to P.

Moreover, we derive the explicit form of the t-component of the BT for all the soliton equations in the hierarchy found by Jaulent and Miodek (JM).

In § 3 we briefly rederive the known result that the Zakharov–Shabat (ZS) and the JJM spectral problems are gauge equivalent (Jaulent and Miodek 1977). The potentials in the two spectral problems are related by Miura-like transformations, i.e. Riccati equations, that, in general, cannot be solved by quadratures. Consequently, those who are interested, for instance, in finding explicit solutions of the JM soliton equations or in building and algebraic procedure for solving approximately the inverse problem, must study directly the JJM spectral problem, which is less symmetric and more involved to manage than the ZS spectral problem.

Moreover, in § 3, we show that this gauge equivalence can be used to derive hidden properties of the two spectral problems and, in § 4, we use the equivalence as a guide in order to derive two new simpler one-parameter BTs for the JJM spectral problem, which we shall call the elementary BTs of the first and second kind. The two-parameter BT found by Laddomada and Tu (1982) (that we call the full BT) can be obtained just by applying successively elementary BTs.

In § 5, by using a general procedure proposed by Boiti *et al* (1983b), we prove the permutability theorem for elementary and full BTs and we write explicitly the corresponding double BT, or superposition formulae, both for elementary and full BTs.

Because the JJM spectral problem is a two-field problem one can define two independent transmission and reflection coefficients. In § 6 we compute how they transform under an elementary BT of the first and second kind and under a full BT. The results are analogous to those for the Schrödinger equation with a λ-independent potential and we conclude that the procedure proposed by Sabatier for exploring the spaces of spectral data and potentials can be extended to the JJM spectral problem (Sabatier 1983c).

In § 7 we compute the explicit solutions of the soliton equations in the JM hierarchy for which the λ-spectrum consists of one and two discrete eigenvalues. With a convenient choice of the constant of integration in the first and in the second case one gets kink-like soliton and bell-like soliton solutions.

2. The full Bäcklund transformation

It is convenient to write (1.1) in spinor form

$$-Y_{xx}+(P+\lambda Q)Y=\lambda^2 Y \tag{2.1}$$

where

$$Y=(y_1, y_2) \tag{2.2}$$

with y_1 and y_2 any two solutions of (1.1) and, successively, by putting

$$\Psi=\begin{pmatrix} Y \\ Y_x \end{pmatrix} \tag{2.3}$$

and

$$U=\sigma_+ +(P+\lambda Q-\lambda^2)\sigma_- \equiv U_0\lambda^2+U_1\lambda+U_2 \tag{2.4}$$

with $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and σ_i $(i=1,2,3)$ the 2×2 Pauli matrices to cast (2.1) into the equivalent form

$$\Psi_x = U\Psi. \tag{2.5}$$

If one is interested in the JM soliton equations related to (2.5) the time dependence of Ψ and, consequently, the time evolution of the fields P and Q are fixed by requiring Ψ to satisfy the so-called auxiliary spectral equation (Ablowitz *et al* 1974)

$$\Psi_t = V\Psi \tag{2.6}$$

with V a polynomial in λ

$$V(P, Q; \lambda) = \sum_{i=-2}^{n} V_j(P, Q)\lambda^{n-j}. \tag{2.7}$$

The integrability condition for the equations (2.5) and (2.6)

$$U_t - V_x + [U, V] = 0 \tag{2.8}$$

is the so-called Lax representation (Lax 1968) for the JM soliton equations. It has been explicitly solved (Laddomada and Tu 1982) furnishing the coefficients V_j and the JM soliton equations.

According to a general procedure (Boiti and Tu 1982, Boiti *et al* 1983a, Levi *et al* 1982), that can be interpreted as a development of the so-called dressing method (Zakharov 1980, Zakharov and Mikhailov 1978, Miklailov 1981), the Bäcklund transformed $\bar{P}$, $\bar{Q}$ of P, Q can be generated by a non-singular 2×2 matrix gauge transformation $B = B(\bar{P}, \bar{Q}, P, Q; \lambda)$ of Ψ.

The gauge transform of Ψ

$$\bar{\Psi} = B\Psi \tag{2.9}$$

is required to satisfy the principal spectral equation

$$\bar{\Psi}_x = \bar{U}\bar{\Psi} \tag{2.10}$$

and the auxiliary spectral equation

$$\bar{\Psi}_t = \bar{V}\bar{\Psi}, \tag{2.11}$$

where $\bar{U}$ and $\bar{V}$ are obtained from U and V, respectively, by substituting in them $\bar{P}$, $\bar{Q}$ for P, Q.

It is easy to verify that B must satisfy the matrix differential equations

$$B_x = \bar{U}B - BU \tag{2.12}$$

$$B_t = \bar{V}B - BV. \tag{2.13}$$

By cross differentiating one gets

$$B_{xt} - B_{tx} = (\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}])B - B(U_t - V_x + [U, V]) \tag{2.14}$$

and, consequently, if the gauge B satisfies (2.12) and (2.13) and P, Q are solutions of the soliton equation defined by the Lax representation

$$U_t - V_x + [U, V] = 0, \tag{2.15}$$

then $\bar{P}$, $\bar{Q}$ are solutions of the same soliton equation induced by

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0. \tag{2.16}$$

For B a polynomial of second order

$$B = B_0\lambda^2 + B_1\lambda + B_2 \tag{2.17}$$

the form of B_0, B_1 and B_2 has been explicitly derived by solving (2.12) and (2.13) (Laddomada and Tu 1982).

The x- and the t-components of the BT generated by the gauge B are obtained by equating the coefficients of λ^0 and λ in (2.12) and (2.13). Precisely, one gets for the x-component

$$B_{2_x} = \bar{U}_2 B_2 - B_2 U_2 \tag{2.18}$$

$$B_{1_x} = \bar{U}_2 B_1 - B_1 U_2 + \bar{U}_1 B_2 - B_2 U_1 \tag{2.19}$$

and for the t-component

$$B_{2_t} = \bar{V}_n B_2 - B_2 V_n \tag{2.20}$$

$$B_{1_t} = \bar{V}_{n-1} B_2 - B_2 V_{n-1} + \bar{V}_n B_1 - B_1 V_n. \tag{2.21}$$

The t-component of the BT has a specific form for any considered equation in the JM hierarchy, while the x-component has a universal character, in the sense that it is unchanged in form for all the equations in the hierarchy. In fact it can be considered as an invariance equation for the principal spectral problem.

The explicit form of the x- and t-components is obtained, respectively, in (Laddomada and Tu 1982) and in (Simone 1983). We are here interested in the x-component, which after one integration, can be written as follows

$$(\bar{Q} + Q)_x \cos\theta + 2(\bar{Q} + Q)(\cos\theta)_x + 2(\bar{P} - P)\sin\theta + (\bar{Q} - Q)\mathscr{I} = 0 \tag{2.22}$$

$$\cos\theta(\cos\theta)_{xx} - \tfrac{1}{2}[(\cos\theta)_x]^2 - (\bar{P} + P)\cos^2\theta + \tfrac{1}{2}\mathscr{I}^2 - 8\alpha_0^2 = 0 \tag{2.23}$$

where

$$\theta = \tfrac{1}{2} I(\bar{Q} - Q) + \theta_0 \tag{2.24}$$

and

$$\mathscr{I} = I[(\bar{P} - P)\cos\theta] + 4\alpha_0. \tag{2.25}$$

(Note the misprint in (15a) of Laddomada and Tu 1982.)

I is the integral operator

$$I[\;\;] = \int_x^{+\infty} \mathrm{d}x[\;\;] \tag{2.26}$$

and α_0, θ_0 are arbitrary constants.

By multiplying equation (2.22) by $\cos\theta$ one gets the ordinary differential equation

$$[(\bar{Q} + Q)\cos^2\theta]_x = 2[\mathscr{I}\sin\theta]_x \tag{2.27}$$

that can be explicitly integrated to

$$\mathscr{I} = \frac{4\alpha_0 \sin\theta_0}{\sin\theta} + \tfrac{1}{2}(\bar{Q} + Q)\frac{\cos^2\theta}{\sin\theta}. \tag{2.28}$$

Therefore, by inserting this $\mathscr{I}$ into (2.22) and (2.23) we get two differential equations local in $I(Q)$ and $I(\bar{Q})$ and algebraic in P, $\bar{P}$.

It is worthwhile noting that in the case $Q \equiv \bar{Q} \equiv 0$, when the JJM spectral equation reduces to the usual one-dimensional Schrödinger equation, θ_0 must be set to zero, (2.22) is trivially satisfied and the BT cannot be obtained in the simplified algebraic form in P and $\bar{P}$.

The matrix coefficients B_0, B_1 and B_2 of the Bäcklund gauge B, by using this simplified version of the BT, can be rewritten in the following form which is local in $I(Q)$ and $I(\bar{Q})$ and independent of P, $\bar{P}$:

$$B_0 = \tfrac{1}{2}\cos\theta\sigma_- \tag{2.29}$$

$$B_1 = -\tfrac{1}{4}(\bar{Q}+Q)\cos\theta\sigma_- + \tfrac{1}{2}\sin\theta\mathbb{1} \tag{2.30}$$

$$B_2 = \tfrac{1}{4}(\cos\theta)_x\sigma_3 + \tfrac{1}{4}\mathcal{I}\mathbb{1} - \tfrac{1}{2}\cos\theta\sigma_+ + \tfrac{1}{8}(\cos\theta)^{-1}\{[(\cos\theta)_x]^2 - \mathcal{I}^2 + 16\alpha_0^2\}\sigma_-. \tag{2.31}$$

3. Gauge equivalence between the Jaulent–Jean–Miodek and the Zakharov–Shabat spectral problems

Let us introduce the function

$$v = \exp[-\mathrm{i}I(Q)] \tag{3.1}$$

and the Miura transformation for P

$$p^2 + p_x = P \tag{3.2}$$

with p tending to zero at infinity.

Then the gauge transformation

$$\Phi = E\Psi \tag{3.3}$$

with

$$E = \frac{\mathrm{i}}{\lambda}\begin{pmatrix} v^{-1/2} & 0 \\ 0 & v^{1/2} \end{pmatrix}\begin{pmatrix} -\mathrm{i}\lambda - p & 1 \\ -\mathrm{i}\lambda + p & -1 \end{pmatrix} \tag{3.4}$$

transforms any matrix solution Ψ of the JJM spectral equation (2.5) into a matrix solution Φ of the Zakharov–Shabat (ZS) spectral problem (Zakharov and Shabat 1979)

$$\Phi_x = U_{\mathrm{ZS}}\Phi \tag{3.5}$$

$$U_{\mathrm{ZS}} = \begin{pmatrix} -\mathrm{i}\lambda & q \\ r & \mathrm{i}\lambda \end{pmatrix}. \tag{3.6}$$

The potentials q and r are related to v and p by the equations

$$v_x = qv^2 - r \tag{3.7}$$

$$2p = qv + rv^{-1}. \tag{3.8}$$

In fact, it is easy to verify directly that

$$U_{\mathrm{ZS}} = E_xE^{-1} + EUE^{-1}. \tag{3.9}$$

Vice versa, once given q and r of a ZS spectral equation, the solution v (tending to 1 at $+\infty$) of the Miura-like transformation (3.7) and p as defined in (3.8) furnish, through (3.1) and (3.2), the potentials P and Q of a gauge equivalent JJM spectral problem.

The equivalence gauge E can be used to transform the Bäcklund gauges B of the JJM spectral problem into the Bäcklund gauges of the ZS spectral problem according to the formula

$$B_{ZS} = g(\lambda)\bar{E}BE^{-1} \tag{3.10}$$

where $\bar{E}$ is obtained from E by substituting $\bar{Q}$, $\bar{P}$ for Q, P and $g(\lambda)$ is an arbitrary function of λ.

We expect the B_{ZS} in (3.10) to be related to the Bäcklund gauges (see Boiti *et al* 1983b)

$$B_{ZS} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\lambda + \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} - \tfrac{1}{2}\mathrm{i}\begin{pmatrix} \alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} I(\bar{q}\bar{r} - qr) + \tfrac{1}{2}\mathrm{i}\begin{pmatrix} 0 & -\alpha_2\bar{q} + \alpha_1 q \\ \alpha_1\bar{r} - \alpha_2 r & 0 \end{pmatrix}. \tag{3.11}$$

The α_i's and β_i's ($i = 1, 2$) are arbitrary constants. The integral term $I(\bar{q}\bar{r} - qr)$ satisfies the algebraic equation

$$-\alpha_1\alpha_2[I(\bar{q}\bar{r} - qr)]^2 + 2\mathrm{i}(\alpha_1\beta_2 - \alpha_2\beta_1)I(\bar{q}\bar{r} - qr) + (\alpha_2\bar{q} - \alpha_1 q)(\alpha_1\bar{r} - \alpha_2 r) = 0 \tag{3.12}$$

and can be transformed into a local term.

The Bäcklund transformed $\bar{q}$, $\bar{r}$ of q, r must be related to $\bar{P}$, $\bar{Q}$ by a version of the equations (3.1), (3.2), (3.7) and (3.8) obtained by substituting in them all quantities with the corresponding barred quantities.

Since the solutions of (2.5) and (3.5) are uniquely determined by their asymptotic behaviour, say at $x = +\infty$, the Bäcklund gauges B and B_{ZS} are uniquely determined by their values at $x = +\infty$. Therefore, in order to identify the B_{ZS} in (3.10) with the B_{ZS} in (3.11), it is convenient to compare them at $x = +\infty$.

The identification is possible in two cases (we exclude the uninteresting case $B_{ZS} \propto \mathbb{1}$).

In the first case

$$\alpha_1 = \alpha_2 = 0 \tag{3.13}$$

and we get ($\beta_1 \neq 0$, $\beta_2 \neq 0$)

$$g(\lambda) = 2(\beta_1\beta_2)^{1/2}\lambda^{-1} \tag{3.14}$$

$$\alpha_0 = 0 \tag{3.15}$$

$$\sin\theta_0 = \tfrac{1}{2}(\beta_1 + \beta_2)\cdot(\beta_1\beta_2)^{-1/2} \tag{3.16}$$

$$\cos\theta_0 = -\tfrac{1}{2}\mathrm{i}(\beta_1 - \beta_2)\cdot(\beta_1\beta_2)^{-1/2}. \tag{3.17}$$

In the second case

$$\beta_1 = \beta_2 = \beta \tag{3.18}$$

and we get ($\alpha_1 \neq 0$, $\alpha_2 \neq 0$)

$$g(\lambda) = 2(\alpha_1\alpha_2)^{1/2} \tag{3.19}$$

$$\alpha_0 = \tfrac{1}{2}\beta(\alpha_1\alpha_2)^{-1/2} \tag{3.20}$$

$$\sin\theta_0 = \tfrac{1}{2}(\alpha_1 + \alpha_2)\cdot(\alpha_1\alpha_2)^{-1/2} \tag{3.21}$$

$$\cos\theta_0 = -\tfrac{1}{2}\mathrm{i}(\alpha_1 - \alpha_2)\cdot(\alpha_1\alpha_2)^{-1/2}. \tag{3.22}$$

It is relevant that the equation (3.10) contains more information than that supplied by the previous relations between parameters of the Bäcklund gauges of the JJM and ZS spectral problems.

In fact, in the second case ($\alpha_0 \neq 0$), by equating the coefficients of the powers of λ in both sides, after some tedious but direct computation, one gets the non-trivial equation

$$(\bar{p}\sin 2\theta + \tfrac{1}{2}Q + \tfrac{1}{2}\bar{Q}\cos 2\theta + 4\alpha_0 \sin\theta_0)\cdot(p\sin 2\theta - \tfrac{1}{2}\bar{Q} - \tfrac{1}{2}Q\cos 2\theta - 4\alpha_0\sin\theta_0)$$
$$= -16\alpha_0^2 \sin^2\theta, \tag{3.23}$$

which furnishes $\bar{p}$ in terms of Q, $\bar{Q}$ and p.

Therefore, if two pairs of fields q, r and P, Q related by the gauge E are known together with the Bäcklund transform $\bar{P}$, $\bar{Q}$ of P, Q, the equation (3.23) solves explicitly the Riccati equation

$$\bar{p}^2 + \bar{p}_x = \bar{P}. \tag{3.24}$$

Consequently, from

$$\bar{v}_x = \bar{q}\bar{v}^2 - \bar{r} \tag{3.25}$$

and

$$2\bar{p} = \bar{q}\bar{v} + \bar{r}\bar{v}^{-1} \tag{3.26}$$

one obtains $\bar{q}$ and $\bar{r}$, the Bäcklund transforms of q and r.

Vice versa, if the equation (3.23) is written in the equivalent form

$$\{[\bar{q}\exp(2\mathrm{i}\theta_0) + q]v^2 + 8\mathrm{i}\alpha_0 v\sin\theta_0 - \bar{r}\exp(-2\mathrm{i}\theta_0) - r\}\{[\bar{q} + q\exp(-2\mathrm{i}\theta_0)]\bar{v}^2$$
$$+ 8\mathrm{i}\alpha_0\bar{v}\sin\theta_0 - \bar{r} - r\exp(2\mathrm{i}\theta_0)\}$$
$$= 16\alpha_0^2[v\exp(\mathrm{i}\theta_0) - \bar{v}\exp(-\mathrm{i}\theta_0)]^2 \tag{3.27}$$

it can be used to derive by a trivial algebraic procedure $\bar{v}$ in terms of q, r, $\bar{q}$, $\bar{r}$ and v.

The solution thus found, $\bar{v}$ of the Riccati equation (3.25), substituted into the equation

$$\mathrm{i}\bar{Q} = \bar{v}_x\bar{v}^{-1}, \tag{3.28}$$

furnishes $\bar{Q}$, while equation (2.22) (or (2.23)) for the BT furnishes $\bar{P}$.

In conclusion we can say that formula (3.23) solves explicitly the BT of the ZS spectral problem once one has solved the BT of the JJM spectral problem (and *vice versa*—equation (3.27)).

In the first case ($\alpha_0 = 0$) the BT for the ZS spectral problem is the trivial rescaling transformation

$$\bar{q} = -q\exp(-2\mathrm{i}\theta_0) \tag{3.29}$$

$$\bar{r} = -r\exp(2\mathrm{i}\theta_0) \tag{3.30}$$

and by equating the coefficients of the powers of λ in (3.10) one gets

$$\bar{p}\sin 2\theta + \tfrac{1}{2}Q + \tfrac{1}{2}\bar{Q}\cos 2\theta = 0 \tag{3.31}$$

$$p\sin 2\theta - \tfrac{1}{2}\bar{Q} - \tfrac{1}{2}Q\cos 2\theta = 0 \tag{3.32}$$

which are just the expressions in curly brackets in (3.23) evaluated at $\alpha_0 = 0$.

In this case, since the BT $\bar{q}$, $\bar{r}$ of q and r are explicitly known, these equations are of no practical use for computing the BT of the ZS spectral problem. Moreover, they cannot be used to derive $\bar{P}$, $\bar{Q}$ once known q, r, $\bar{q}$, $\bar{r}$ and Q, P because the equivalent version of (3.31) and (3.32) in terms of v and $\bar{v}$ are trivially satisfied, as one can see just by looking at the formulae in curly brackets in (3.27) evaluated at $\alpha_0=0$.

However, we shall show in § 5 that these equations can be used to find a nonlinear superposition formula in the JJM case.

4. Elementary Bäcklund transformations

The equivalence between the ZS and JJM spectral problems and the existence of the so-called elementary Bäcklund transformations (Konopelchenko 1982, Calogero and Degasperis 1983) in the ZS case suggest the search for elementary BTs also in the JJM case.

In the ZS case we call elementary BT of the first kind, the BT generated by the Bäcklund gauge obtained by choosing

$$\alpha_1=0, \qquad \alpha_2\neq 0 \tag{4.1}$$

$$\beta_1=\beta_2=\beta, \tag{4.2}$$

the elementary BT of the second kind, the BT obtained by choosing

$$\alpha_1\neq 0, \qquad \alpha_2=0 \tag{4.3}$$

$$\beta_1=\beta_2=\beta \tag{4.4}$$

and elementary BT of the third kind the BT with

$$\alpha_1=\alpha_2=0 \tag{4.5}$$

$$\beta_1\neq\beta_2. \tag{4.6}$$

The corresponding Bäcklund gauges for the JJM spectral problem will be indicated, respectively, by B^{I}, B^{II} and B^{III}.

Let us rewrite the equations (3.20)–(3.22) in the following way

$$\frac{\sin\theta_0}{\alpha_0}=-\frac{1}{\lambda_0}-\frac{\lambda_0}{4\alpha_0^2} \tag{4.7}$$

$$\frac{\cos\theta_0}{\alpha_0}=-\mathrm{i}\frac{1}{\lambda_0}+\mathrm{i}\frac{\lambda_0}{4\alpha_0^2} \tag{4.8}$$

with

$$\lambda_0=-\beta(\alpha_2)^{-1} \tag{4.9}$$

$$\alpha_0=\tfrac{1}{2}\beta(\alpha_1\alpha_2)^{-1/2} \tag{4.10}$$

or, alternatively,

$$\frac{\sin\theta_0}{\alpha_0}=-\frac{1}{\mu_0}-\frac{\mu_0}{4\alpha_0^2} \tag{4.11}$$

$$\frac{\cos\theta_0}{\alpha_0}=\mathrm{i}\frac{1}{\mu_0}-\mathrm{i}\frac{\mu_0}{4\alpha_0^2} \tag{4.12}$$

with

$$\mu_0 = -\beta(\alpha_1)^{-1} \tag{4.13}$$

$$\alpha_0 = \tfrac{1}{2}\beta(\alpha_1\alpha_2)^{-1/2}. \tag{4.14}$$

The Bäcklund gauges B^{I} and B^{II} are obtained by taking the limit for α_0, $\sin\theta_0$ and $\cos\theta_0$ going to infinity at fixed λ_0 using (4.7), (4.8) and, respectively, at fixed μ_0 using (4.11), (4.12).

The Bäcklund gauge B^{III} is obtained by taking $\alpha_0 = 0$.

Any Bäcklund gauge relative to the JJM spectral problem, which is a polynomial in λ, can be obtained by applying successively elementary Bäcklund gauges.

In particular, the full Bäcklund gauge B of § 3 ($\alpha_0 \neq 0$) can be obtained by applying successively a B^{I} and a B^{II}. This can be easily proved by showing that with a convenient choice of the parameters the value at $x=+\infty$ of $B^{\mathrm{I}}B^{\mathrm{II}}$ coincides with the value at $x=+\infty$ of B.

We get for B^{I}

$$\begin{aligned} B^{\mathrm{I}} = -\tfrac{1}{2}\mathrm{i}\lambda_0^{-1}\exp(\mathrm{i}\tau)\{&\sigma_-\lambda^2 - [\mathrm{i}\mathbb{1} + \tfrac{1}{2}(\bar{Q}+Q)\sigma_-]\lambda \\ &+[2\mathrm{i}\lambda_0\exp(-2\mathrm{i}\tau) + \tfrac{1}{4}\mathrm{i}\bar{Q}+Q)]\mathbb{1} - \tfrac{1}{4}\mathrm{i}(\bar{Q}-Q)\sigma_3 - \sigma_+ \\ &+[\tfrac{1}{4}\bar{Q}Q - 8\mathrm{i}\lambda_0^2\exp(-3\mathrm{i}\tau)\sin\tau + \lambda_0(\bar{Q}+Q)\exp(-2\mathrm{i}\tau)]\sigma_-\} \end{aligned} \tag{4.15}$$

and for the corresponding BT

$$\mathrm{i}(\bar{Q}+Q)_x = -2(\bar{P}-P) - \tfrac{1}{2}(\bar{Q}^2-Q^2) + 4\lambda_0(\bar{Q}-Q)\exp(-2\mathrm{i}\tau) \tag{4.16}$$

$$\mathrm{i}(\bar{Q}-Q)_x = -2(\bar{P}+P) - \tfrac{1}{2}(\bar{Q}^2+Q^2) + 32\mathrm{i}\lambda_0^2\exp(-3\mathrm{i}\tau)\sin\tau - 4\lambda_0(\bar{Q}+Q)\exp(-2\mathrm{i}\tau), \tag{4.17}$$

while for B^{II} we get

$$\begin{aligned} B^{\mathrm{II}} = \tfrac{1}{2}\mathrm{i}\mu_0^{-1}\exp(-\mathrm{i}\tau)\{&\sigma_-\lambda^2 + [\mathrm{i}\mathbb{1} - \tfrac{1}{2}(\bar{Q}+Q)\sigma_-]\lambda \\ &-[2\mathrm{i}\mu_0\exp(2\mathrm{i}\tau) + \tfrac{1}{4}\mathrm{i}(\bar{Q}+Q)]\mathbb{1} + \tfrac{1}{4}\mathrm{i}(\bar{Q}-Q)\sigma_3 - \sigma_+ \\ &+[\tfrac{1}{4}\bar{Q}Q + 8\mathrm{i}\mu_0^2\exp(3\mathrm{i}\tau)\sin\tau + \mu_0(\bar{Q}+Q)\exp(2\mathrm{i}\tau)]\sigma_-\} \end{aligned} \tag{4.18}$$

and for the corresponding BT

$$\mathrm{i}(\bar{Q}+Q)_x = 2(\bar{P}-P) + \tfrac{1}{2}(\bar{Q}^2-Q^2) - 4\mu_0(\bar{Q}-Q)\exp(2\mathrm{i}\tau) \tag{4.19}$$

$$\mathrm{i}(\bar{Q}-Q)_x = 2(\bar{P}+P) + \tfrac{1}{2}(\bar{Q}^2+Q^2) + 32\mathrm{i}\mu_0^2\exp(3\mathrm{i}\tau)\sin\tau + 4\mu_0(\bar{Q}+Q)\exp(2\mathrm{i}\tau). \tag{4.20}$$

In both cases τ is defined as follows

$$\tau = \tfrac{1}{2}I(\bar{Q}-Q). \tag{4.21}$$

It is important to note that the difference between the equations defining the BT of the first kind can be cast into the form of a Riccati equation

$$-\rho_x - \rho^2 + P + \lambda_0 Q - \lambda_0^2 = 0 \tag{4.22}$$

with

$$\rho = 2\mathrm{i}\lambda_0\exp(-2\mathrm{i}\tau) + \tfrac{1}{2}\mathrm{i}Q - \mathrm{i}\lambda_0. \tag{4.23}$$

The solution ρ of the Riccati equation (4.22) furnishes $\bar{Q}$ and $\bar{P}$ by means of the formulae

$$\bar{Q} = Q - \mathrm{i}(\mathrm{d}/\mathrm{d}x)\log(\rho - \tfrac{1}{2}\mathrm{i}Q + \mathrm{i}\lambda_0) \tag{4.24}$$

$$\bar{P} = P - \rho_x - \tfrac{1}{2}\mathrm{i}Q_x - \tfrac{1}{2}(\mathrm{d}^2/\mathrm{d}x^2)\log(\rho - \tfrac{1}{2}\mathrm{i}Q + \mathrm{i}\lambda_0) + \tfrac{1}{2}\mathrm{i}Q(\mathrm{d}/\mathrm{d}x)\log(\rho - \tfrac{1}{2}\mathrm{i}Q + \mathrm{i}\lambda_0) + \tfrac{1}{4}[(\mathrm{d}/\mathrm{d}x)\log(\rho - \tfrac{1}{2}\mathrm{i}Q + \mathrm{i}\lambda_0)]^2. \tag{4.25}$$

Analogously the BT of the second kind can be written as follows

$$-\sigma_x - \sigma^2 + P + \mu_0 Q - \mu_0^2 = 0 \tag{4.26}$$

with

$$\sigma = -2\mathrm{i}\mu_0\exp(2\mathrm{i}\tau) - \tfrac{1}{2}\mathrm{i}Q + \mathrm{i}\mu_0. \tag{4.27}$$

The solution σ of the Riccati equation (4.26) furnishes $\bar{Q}$ and $\bar{P}$ via the formulae

$$\bar{Q} = Q + \mathrm{i}(\mathrm{d}/\mathrm{d}x)\log(\sigma + \tfrac{1}{2}\mathrm{i}Q - \mathrm{i}\mu_0) \tag{4.28}$$

$$\bar{P} = P - \sigma_x + \tfrac{1}{2}\mathrm{i}Q_x - \tfrac{1}{2}(\mathrm{d}^2/\mathrm{d}x^2)\log(\sigma + \tfrac{1}{2}\mathrm{i}Q - \mathrm{i}\mu_0) - \tfrac{1}{2}\mathrm{i}Q(\mathrm{d}/\mathrm{d}x)\log(\sigma + \tfrac{1}{2}\mathrm{i}Q - \mathrm{i}\mu_0) + \tfrac{1}{4}[(\mathrm{d}/\mathrm{d}x)\log(\sigma + \tfrac{1}{2}\mathrm{i}Q - \mathrm{i}\mu_0)]^2. \tag{4.29}$$

We conclude that both elementary Bäcklund transformations can be cast into a form that can be considered the generalisation of the Darboux transformation to the case in which the potential in the Schrödinger equation depends linearly on the spectral parameter λ.

5. The double Bäcklund transformations

The composition of two Bäcklund gauges $B(Q_3, Q_1; \alpha_0^{(2)}, \theta_0^{(2)}) \cdot B(Q_1, Q_0; \alpha_0^{(1)}, \theta_0^{(1)})$—or, for short, $B_{31}^{(2)}B_{10}^{(1)}$, where the subscripts refer to the fields and the superscripts to the parameters involved—is a Bäcklund gauge that transforms the field Q_0 into the field Q_3. This can be verified by computing its x- and t-derivative by using (2.12) and (2.13) written successively for $B_{31}^{(2)}$ and $B_{10}^{(1)}$.

We shall call $B_{31}^{(2)}B_{10}^{(1)}$ a double Bäcklund gauge and its corresponding BT a double BT.

The Bäcklund gauges satisfy the so-called permutability theorem

$$B_{31}^{(2)}B_{10}^{(1)} = B_{32}^{(1)}B_{20}^{(2)}. \tag{5.1}$$

Because the Bäcklund gauges are uniquely determined by their asymptotic value at $x = +\infty$, the theorem (5.1) follows from the corresponding trivial identity at $x = +\infty$.

The permutability theorem can be expressed by saying that the diagram of figure 1 commutes. Arrows represent BTs.

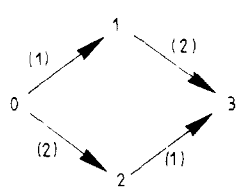

Figure 1.

It is convenient to look first at the simpler case in which all the Bäcklund gauges involved are elementary Bäcklund gauges. We consider, for instance, the case represented in the diagram of figure 2, where the BTs of the first and second kind are represented, respectively, by single and double lines and the parameters of the BTs are explicilty indicated.

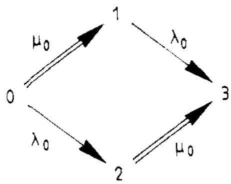

Figure 2.

It is easy to verify that a BT of the first kind (second) with parameter λ_0 (μ_0) is the inverse of a BT of the second kind (first) with parameter λ_0 (μ_0).

Consequently, the previous commuting diagram can be equivalently drawn as in figures 3–5.

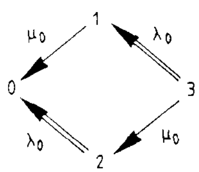

Figure 3.

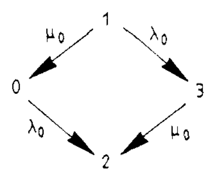

Figure 4.

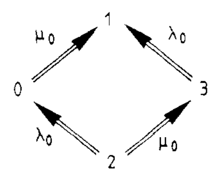

Figure 5.

We expect to get from (5.1), by identifying the powers of λ on both sides, four nonlinear superposition formulae giving, respectively, Q_3 or Q_0 or Q_2 or Q_1 in terms of the remaining fields as suggested by the four commuting diagrams in figures 2–5.

In fact we get

$$\exp(-2\mathrm{i}\tau_{30})=\frac{\mu_0}{\lambda_0}+2\frac{\mu_0}{\lambda_0}(\lambda_0-\mu_0)[2\lambda_0\exp(-2\mathrm{i}\tau_{20})+2\mu_0\exp(2\mathrm{i}\tau_{10})+Q_0-2\lambda_0]^{-1} \tag{5.2}$$

$$\exp(2\mathrm{i}\tau_{30})=\frac{\lambda_0}{\mu_0}-2\frac{\lambda_0}{\mu_0}(\lambda_0-\mu_0)[2\lambda_0\exp(-2\mathrm{i}\tau_{31})+2\mu_0\exp(2\mathrm{i}\tau_{32})+Q_3-2\mu_0]^{-1} \tag{5.3}$$

$$\exp(-2\mathrm{i}\tau_{21})=\frac{2\lambda_0\exp(-2\mathrm{i}\tau_{31})-2\mu_0\exp(2\mathrm{i}\tau_{10})+Q_3-Q_0}{2\lambda_0\exp(-2\mathrm{i}\tau_{10})-2\mu_0\exp(2\mathrm{i}\tau_{31})} \tag{5.4}$$

$$\exp(2\mathrm{i}\tau_{21})=\frac{2\mu_0\exp(2\mathrm{i}\tau_{32})-2\lambda_0\exp(-2\mathrm{i}\tau_{20})+Q_3-Q_0}{2\mu_0\exp(2\mathrm{i}\tau_{20})-2\lambda_0\exp(-2\mathrm{i}\tau_{32})} \tag{5.5}$$

where

$$\tau_{ij}=\tfrac{1}{2}I(Q_i-Q_j). \tag{5.6}$$

If the Q_i ($i=0,1,2,3$) are considered as independent fields it is easy to show that only two superposition formulae are independent. We can choose, for instance, the first two.

If one applies the permutability theorem (5.1) to the full Bäcklund gauge $B(\bar{Q}, Q; \alpha_0, \theta_0)$ defined in (2.29)–(2.31), the computations become very complicated and tedious (a preliminary study has been made by Simone 1983). However, the resulting superposition formula obtained by identifying the coefficients of the powers of λ in (5.1) can be cast in the following rather symmetric form

$$\exp(-2\mathrm{i}\tau_{30}) = \exp(2\mathrm{i}\theta_0^{(1)} + 2\mathrm{i}\theta_0^{(2)}) \frac{(A+B)^2 + C}{(A+B')^2 + C'} \tag{5.7}$$

where

$$A = \tfrac{1}{8}[Q_1 \sin(2\theta_2) - Q_2 \sin(2\theta_1) + Q_0 \sin(2\theta_2 - 2\theta_1)] \tag{5.8}$$

$$B = \alpha_0^{(1)} \sin\theta_0^{(1)} \cos\theta_1 \sin(2\theta_2) \exp(-\mathrm{i}\theta_1) - (1 \leftrightarrow 2) \tag{5.9}$$

$$C = [\alpha_0^{(1)} \sin\theta_0^{(1)} \sin\theta_1 \sin(2\theta_2) \exp(-\mathrm{i}\theta_1) - (1 \leftrightarrow 2)]^2 \\ - 4\mathrm{i}[\alpha_0^{(1)^2} \cos\theta_2 \sin^2\theta_1 \sin^2\theta_2 \sin(\theta_2 - \theta_1) \exp(-\mathrm{i}\theta_1) + (1 \leftrightarrow 2)] \tag{5.10}$$

with

$$\theta_2 = \tfrac{1}{2} I(Q_2 - Q_0) + \theta_0^{(2)} \tag{5.11}$$

$$\theta_1 = \tfrac{1}{2} I(Q_1 - Q_0) + \theta_0^{(1)} \tag{5.12}$$

B' and C' are obtained from B and C, respectively, changing i into $-$i.

All the previously obtained superposition formulae can be derived, as special cases of this formula, by computing the appropriate limit for $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ going to infinity.

For instance, by writing

$$\sin\theta_0^{(1)} \cdot (\alpha_0^{(1)})^{-1} = -\mu_0^{-1} - \tfrac{1}{4}\mu_0(\alpha_0^{(1)})^{-2} \tag{5.13}$$

$$\cos\theta_0^{(1)} \cdot (\alpha_0^{(1)})^{-1} = \mathrm{i}\mu_0^{-1} - \tfrac{1}{4}\mathrm{i}\mu_0(\alpha_0^{(1)})^{-2} \tag{5.14}$$

and

$$\sin\theta_0^{(2)} \cdot (\alpha_0^{(2)})^{-1} = -\lambda_0^{-1} - \tfrac{1}{4}\lambda_0(\alpha_0^{(2)})^{-2} \tag{5.15}$$

$$\cos\theta_0^{(2)} \cdot (\alpha_0^{(2)})^{-1} = -\mathrm{i}\lambda_0^{-1} + \tfrac{1}{4}\mathrm{i}\lambda_0(\alpha_0^{(2)})^{-2} \tag{5.16}$$

and by taking the limit $\alpha_0^{(1)} \to \infty$ and $\alpha_0^{(2)} \to \infty$ one gets the superposition formula (5.2). The second independent superposition formula (5.3) can be obtained simply by noting that the diagram in figure 3 is the same as the diagram in figure 2 once the fields and the parameters of the BTs have been renamed appropriately.

Formula (5.7) can also be used to get the superposition formula related to the composition of two B^{III}, of a B^{III} with a B^{I} and a B^{III} with a B^{II}.

In the first case, because at $\alpha_0^{(1)} = \alpha_0^{(2)} = 0$ the numerator and the denominator in the right-hand side of (5.7) are equal and because, in general, $\exp(-2\mathrm{i}\tau_{30}) \neq \exp(2\mathrm{i}\theta_0^{(1)} + 2\mathrm{i}\theta_0^{(2)})$, we deduce that they must be zero. More precisely, we get

$$Q_0 \sin(2\tau_{21} + 2\theta_0^{(2)} - 2\theta_0^{(1)}) + Q_1 \sin(2\tau_{20} + 2\theta_0^{(2)}) - Q_2 \sin(2\tau_{10} + 2\theta_0^{(1)}) = 0. \tag{5.17}$$

It is worthwhile noting that this equation can be obtained directly from (3.32) written for the two BTs generated by B_{10}^{III} and B_{20}^{III}.

By taking into account that $B(\bar{Q}, Q; 0, \theta_0)$ is gauge equivalent to the $B_{\mathrm{ZS}} = \frac{1}{2}(\mathbb{1} + \sigma_3)\beta_1 + \frac{1}{2}(\mathbb{1} - \sigma_3)\beta_2$, whose inverse is $B_{\mathrm{ZS}}^{-1} = \frac{1}{2}(\mathbb{1} + \sigma_3)\beta_1^{-1} + \frac{1}{2}(\mathbb{1} - \sigma_3)\beta_2^{-1}$, and the formulae (3.16), (3.17) relating θ_0 to β_1 and β_2, it is easy to prove that

$$B^{-1}(\bar{Q}, Q; 0, \theta_0) = B(Q, \bar{Q}; 0, -\theta_0 + \pi). \tag{5.18}$$

Consequently, the two diagrams in figures 6 and 7, in which a triple line indicates a BT of the third kind, relate the same fields.

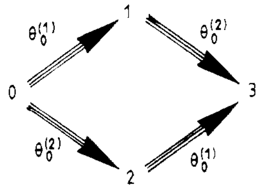

Figure 6.

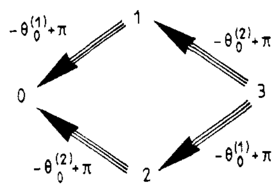

Figure 7.

Therefore from (5.17) applied to the second diagram we get the second independent superposition formula relative to the double BT generated by composing two B^{III},

$$Q_3\sin(2\tau_{12}+2\theta_0^{(1)}-2\theta_0^{(2)})+Q_2\sin(2\tau_{13}-2\theta_0^{(2)})-Q_1\sin(2\tau_{23}-2\theta_0^{(1)})=0. \tag{5.19}$$

In a similar way one can get the superposition formulae relative to the double BT generated by the gauges $B^{\mathrm{III}}B^{\mathrm{I}}$ and $B^{\mathrm{III}}B^{\mathrm{II}}$.

6. Bäcklund transformations of the spectral data

Let us consider the JJM spectral problem in the equivalent form (2.5) of a first-order linear spectral problem and define right F^+ and left F^- matrix Jost solutions (λ real) by

$$F^{\pm}(x,\lambda)\to W(x,\lambda) \qquad \text{as } x\to\pm\infty \tag{6.1}$$

where

$$W(x,\lambda)=\begin{pmatrix}\exp(-\mathrm{i}\lambda x) & \exp(\mathrm{i}\lambda x)\\ -\mathrm{i}\lambda\exp(-\mathrm{i}\lambda x) & \mathrm{i}\lambda\exp(\mathrm{i}\lambda x)\end{pmatrix}. \tag{6.2}$$

P and Q are supposed to vanish at infinity sufficiently fast.

The 'scattering matrix'

$$S(\lambda)=\begin{pmatrix}a(\lambda) & c(\lambda)\\ b(\lambda) & d(\lambda)\end{pmatrix} \tag{6.3}$$

relates the right matrix Jost solution to the left matrix Jost solution

$$F^-=F^+S \tag{6.4}$$

and it satisfies the 'unitarity relation'

$$\det S=1. \tag{6.5}$$

In the language of Jaulent and Miodek (1976)

$$a(\lambda)=1/T^+(\lambda) \tag{6.6}$$

$$b(\lambda)=R^+(\lambda)/T^+(\lambda) \tag{6.7}$$

$$c(\lambda)=R^-(-\lambda)/T^-(-\lambda) \tag{6.8}$$

$$d(\lambda)=1/T^-(-\lambda). \tag{6.9}$$

The right and left matrix Jost solutions of the JJM spectral problem relative to the potentials $\bar{P}$, $\bar{Q}$, Bäcklund transforms of P and Q via the gauge B, are given by

$$\bar{F}^{+} = BF^{+}W^{-1}(B^{+})^{-1}W \tag{6.10}$$

$$\bar{F}^{-} = BF^{-}W^{-1}(B^{-})^{-1}W \tag{6.11}$$

where B^{+} and B^{-} are respectively the asymptotic values of B at $x=+\infty$ and at $x=-\infty$.

In fact BF^{+} and BF^{-} are solutions of the Bäcklund transformed spectral problem

$$\bar{\Psi}_x = \bar{U}\bar{\Psi} \tag{6.12}$$

and $W^{-1}(B^{+})^{-1}W$, $W^{-1}(B^{-})^{-1}W$ are matrices independent of x, chosen in such a way as to guarantee the required asymptotic behaviour of $\bar{F}^{\pm}$.

Therefore, the scattering matrix $\bar{S}$ defined by

$$\bar{F}^{-} = \bar{F}^{+}\bar{S} \tag{6.13}$$

is related to the old scattering matrix S by the equation

$$\bar{S} = W^{-1}B^{+}WSW^{-1}(B^{-})^{-1}W. \tag{6.14}$$

Let us consider first the elementary Bäcklund gauges of the first and second kind. In both cases from the second equation (4.17) and (4.20) in the x-component of the BT evaluated at $x=-\infty$, it is easy to derive that

$$\int_{-\infty}^{+\infty} (\bar{Q}-Q)\,\mathrm{d}x = 2k\pi \tag{6.15}$$

with $k\in\mathbb{Z}$ to be determined. Consequently, in both cases $B^{+}=(-1)^{k}B^{-}$.

In the first case (BT of first kind) we get

$$\bar{a}(\lambda) = (-1)^{k}a(\lambda) \tag{6.16}$$

$$\bar{b}(\lambda) = (-1)^{k}b(\lambda)(1-\lambda/\lambda_0) \tag{6.17}$$

$$\bar{c}(\lambda) = (-1)^{k}c(\lambda)(1-\lambda/\lambda_0)^{-1} \tag{6.18}$$

$$\bar{d}(\lambda) = (-1)^{k}d(\lambda) \tag{6.19}$$

or, in terms of the transmission and reflection coefficients,

$$\bar{T}^{\pm}(\pm\lambda) = (-1)^{k}T^{\pm}(\pm\lambda) \tag{6.20}$$

$$\bar{R}^{\pm}(\pm\lambda) = R^{\pm}(\pm\lambda)(1-\lambda/\lambda_0)^{\pm 1} \tag{6.21}$$

with all signs $\pm$ taken at the same level.

In the case of the BT of the second kind, it results that

$$\bar{a}(\lambda) = (-1)^{k}a(\lambda) \tag{6.22}$$

$$\bar{b}(\lambda) = (-1)^{k}b(\lambda)(1-\lambda/\mu_0)^{-1} \tag{6.23}$$

$$\bar{c}(\lambda) = (-1)^{k}c(\lambda)(1-\lambda/\mu_0) \tag{6.24}$$

$$\bar{d}(\lambda) = (-1)^{k}d(\lambda) \tag{6.25}$$

or

$$\bar{T}^{\pm}(\pm\lambda) = (-1)^{k}T^{\pm}(\pm\lambda) \tag{6.26}$$

$$\bar{R}^{\pm}(\pm\lambda) = R^{\pm}(\pm\lambda)(1-\lambda/\mu_0)^{\mp 1}. \tag{6.27}$$

In the case of the BT of the third kind we have no *a priori* information on

$$\tau_0 = \tfrac{1}{2} \int_{-\infty}^{+\infty} (\bar{Q} - Q)\, dx \tag{6.28}$$

and we get

$$\bar{T}^{\pm}(\pm\lambda) = T^{\pm}(\pm\lambda) \exp(\mp i\tau_0) \tag{6.29}$$

$$\bar{R}^{\pm}(\pm\lambda) = -R^{\pm}(\pm\lambda) \exp(\pm 2i\theta_0). \tag{6.30}$$

It is worthwhile noting that according to formulae (6.20) and (6.26) (apart from a possible sign) the two transmission coefficients $T^{\pm}$ do not change after a BT of the first or second kind. Therefore, these BTs are not able to add a soliton to the given solution.

In fact, we shall see in § 7 that the solutions corresponding to a scattering matrix with zero reflection coefficients and transmission coefficient $T^{+}(\lambda)$ (or $T^{-}(\lambda)$) with a pole in the upper λ-plane do not vanish at $x = \pm\infty$.

However, it may occur (see § 7) that solutions corresponding to two values of λ in the discrete spectrum, one giving a pole to $T^{+}(\lambda)$ and the other to $T^{-}(\lambda)$, have the required vanishing behaviour at $x = \pm\infty$.

These two poles can be obtained by using a BT generated by the full Bäcklund gauge $B(\bar{Q}, Q; \alpha_0, \theta_0)$ where

$$\alpha_0 = \tfrac{1}{2}(\lambda_0 \mu_0)^{1/2} \tag{6.31}$$

$$\sin\theta_0 = -\tfrac{1}{2}(\lambda_0 + \mu_0)(\lambda_0\mu_0)^{-1/2} \tag{6.32}$$

$$\cos\theta_0 = \tfrac{1}{2} i(\lambda_0 - \mu_0)(\lambda_0\mu_0)^{-1/2} \tag{6.33}$$

with $\operatorname{Im}\lambda_0 < 0$ and $\operatorname{Im}\mu_0 > 0$.

In this case the value of τ_0 can be inferred by looking at the value of (2.23) at $x = -\infty$. It results that

$$\mathscr{I}(-\infty) = \pm 4\alpha_0 \tag{6.34}$$

and, consequently, from (2.28) we get two possible cases:

$$\tau_0 = k\pi, \qquad k \in \mathbb{Z} \tag{6.35}$$

and

$$\tau_0 = -2\theta_0 + k\pi, \qquad k \in \mathbb{Z}. \tag{6.36}$$

By following a procedure analogous to that followed for the elementary BTs we get that the matrix elements of S transform according to the following formulae, in the first case,

$$\bar{a}(\lambda) = (-1)^k a(\lambda) \tag{6.37}$$

$$\bar{b}(\lambda) = (-1)^k b(\lambda)(\mu_0/\lambda_0)(\lambda - \lambda_0)(\lambda - \mu_0)^{-1} \tag{6.38}$$

$$\bar{c}(\lambda) = (-1)^k c(\lambda)(\lambda_0/\mu_0)(\lambda - \mu_0)(\lambda - \lambda_0)^{-1} \tag{6.39}$$

$$\bar{d}(\lambda) = (-1)^k d(\lambda) \tag{6.40}$$

and, in the second case,

$$\bar{a}(\lambda) = (-1)^{k+1} a(\lambda)(\lambda_0/\mu_0)(\lambda - \mu_0)(\lambda - \lambda_0)^{-1} \tag{6.41}$$

$$\bar{b}(\lambda) = (-1)^{k+1} b(\lambda) \tag{6.42}$$

$$\bar{c}(\lambda) = (-1)^{k+1} c(\lambda) \tag{6.43}$$

$$\bar{d}(\lambda) = (-1)^{k+1} d(\lambda)(\mu_0/\lambda_0)(\lambda - \lambda_0)(\lambda - \mu_0)^{-1}. \tag{6.44}$$

In terms of the transmission and reflection coefficients we get, in the first case,

$$\bar{T}^{\pm}(\pm\lambda) = (-1)^k T^{\pm}(\pm\lambda) \tag{6.45}$$

$$\bar{R}^{\pm}(\pm\lambda) = R^{\pm}(\pm\lambda)\{(\mu_0/\lambda_0)(\lambda - \lambda_0)(\lambda - \mu_0)^{-1}\}^{\pm 1} \tag{6.46}$$

and, in the second case,

$$\bar{T}^{\pm}(\pm\lambda) = (-1)^{k+1} T^{\pm}(\pm\lambda)\{(\mu_0/\lambda_0)(\lambda - \lambda_0)(\lambda - \mu_0)^{-1}\}^{\pm 1} \tag{6.47}$$

$$\bar{R}^{\pm}(\pm\lambda) = R^{\pm}(\pm\lambda)\{(\mu_0/\lambda_0)(\lambda - \lambda_0)(\lambda - \mu_0)^{-1}\}^{\pm 1}. \tag{6.48}$$

7. The soliton solutions

We call solitons the solutions of the evolution equations in the JM hierarchy for which the λ-spectrum consists of discrete eigenvalues. According to a well known procedure they can be obtained by applying recursively the elementary Bäcklund transformation to the trivial solution $Q \equiv P \equiv O$.

Because the BTs are algebraic in $\bar{P}$ and P we shall write explicitly in the following only the Bäcklund transformed field $\bar{Q}$.

The BTs are local in $I(\bar{Q})$ and, consequently, we can enlarge the set of admitted $\bar{Q}$ solutions by including the $\bar{Q}$'s which do not vanish at $x = -\infty$.

By applying the elementary BT of the second kind and of the first kind to $Q \equiv P \equiv O$ one gets, respectively, the solutions Q_1 and Q_2 corresponding to one discrete eigenvalue μ_0 and λ_0 ($\mathrm{Im}\,\mu_0 > 0$, $\mathrm{Im}\,\lambda_0 < 0$)

$$Q_1 = \mu_0 \exp[\mathrm{i}\mu_0(x - \xi_1)]/\cosh[\mathrm{i}\mu_0(x - \xi_1)] \tag{7.1}$$

$$Q_2 = \lambda_0 \exp[-\mathrm{i}\lambda_0(x - \xi_2)]/\cosh[\mathrm{i}\lambda_0(x - \xi_2)]. \tag{7.2}$$

The time evolution of $\xi_1 = \xi_1(t)$ and $\xi_2 = \xi_2(t)$ must be determined according to the specific evolution equation in the JM hierarchy considered. For those evolution equations for which $\mathrm{Im}\,\xi_i = 0$ ($i = 1, 2$) the solitons found are regular at any x and t with a kink-like behaviour.

The soliton solution Q_3 corresponding to two discrete eigenvalue μ_0, λ_0 ($\mathrm{Im}\,\mu_0 > 0$, $\mathrm{Im}\,\lambda_0 < 0$) is obtained by using the nonlinear superposition formula (5.2)

$$\mathrm{e}^{-2\mathrm{i}\tau_{30}} = \mathrm{e}^{2\mathrm{i}\theta_0} \frac{\cosh(\gamma - \mathrm{i}\theta_0) + \mathrm{e}^{-\delta}\cosh(\mathrm{i}\theta_0)}{\cosh(\gamma + \mathrm{i}\theta_0) + \mathrm{e}^{\delta}\cosh(\mathrm{i}\theta_0)}. \tag{7.3}$$

α_0 and θ_0 are defined in (6.31)-(6.33), and

$$\gamma = \mathrm{i}(\lambda_0 - \mu_0)x - \mathrm{i}\lambda_0\xi_1 + \mathrm{i}\mu_0\xi_2 \tag{7.4}$$

$$\delta = -\mathrm{i}(\lambda_0 + \mu_0)x + \mathrm{i}\lambda_0\xi_1 + \mathrm{i}\mu_0\xi_2. \tag{7.5}$$

This soliton solution vanishes at $x = \pm\infty$ and is regular at any x and t for μ_0 and λ_0 located on the imaginary axis.

The transmission coefficients $T^{\pm}(\lambda)$ have a pole respectively at $\lambda = \mu_0$ and at $\lambda = \lambda_0$.

In fact, from (7.3), evaluated at $x = -\infty$, it is easy to show that

$$\tau_0 = -2\theta_0 + k\pi \tag{7.6}$$

with $k \in \mathbb{Z}$ to be determined and, consequently, the spectral data transform as in the second case discussed in § 6.

References

Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Studies Appl. Math.* **53** 249
Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform (SIAM)*
Boiti M, Laddomada C, Pempinelli F and Tu G Z 1983a *Physica* D **9** 425
Boiti M, Pempinelli F and Tu G Z 1983b *Phys. Lett.* **93**A 107
Boiti M and Tu G Z 1982 *Nuovo Cimento* **71B** 253
Calogero F and Degasperis A 1982 *Spectral Transform and Solitons* (Amsterdam: North-Holland)
—— 1983 *Elementary Bäcklund transformations, nonlinear superposition formulae and algebraic construction of solutions for the nonlinear evolution equations solvable by the Zakharov-Shabat spectral transform, Preprint, Roma*
Jaulent M 1976 *J. Math. Phys.* **17** 1351
Jaulent M and Jean C 1976a *Ann. Inst. Henri Poincaré* **25** 105
—— 1976b *Ann. Inst. Henri Poincaré* **25** 119
Jaulent M and Miodek I 1976 *Lett. Math. Phys.* **1** 243
—— 1977 *Lett. Nuovo Cimento* **20** 655
Konopelchenko B G 1982 *Phys. Lett.* **87A** 445
Laddomada C and Tu G Z 1982 *Lett. Math. Phys.* **6** 453
Lax P D 1968 *Commun. 'Pure Appl. Math.* **21** 467
Levi D, Ragnisco O and Sym A 1982 *Lett. Nuovo Cimento* **33** 401
Mikhailov A V 1981 *Physica* **3D** 73
Sabatier P C 1983a *Nuovo Cimento* **78B** 235
—— 1983b *Rational reflection coefficients in one dimensional inverse scattering and applications, Proc. Tulsa Conf. on Scattering Theory* to be published by SIAM
—— 1983c *Critical analysis of the mathematical methods used in electromagnetic inverse theories: a quest for new routes in the space of parameters, Preprint, Montpellier*
Simone M M 1983 *Master Thesis* University of Lecce
Zakharov V E 1980 in *Solitons* ed R K Bullough and P J Caudrey (Berlin: Springer) pp 243–84
Zakharov V E and Mikhailov A V 1978 *Sov. Phys.-JETP* **47** 1017
Zakharov V E and Shabat A B 1979 *Funct. Anal. Priloz.* **13** 166